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Seismometers are insensitive to rotary motion

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Abstract

In the context of seismic observations in the nearfield of the source or when observing the motion of structures the response of pendulum seismometers to rotary motion is discussed occasionally. However, seismometers are insensitive to rotary motion if the output signal is referred to the location of the point mass of the equivalent simple pendulum. The observation of rotary motion deserves specific sensors that are insensitive to linear acceleration.

Rigid-body motion

Seismometers are intended to record the displacement of the ground. The output signal of the seismometer thus must be attributed to the motion of one point. The choice of this location is arbitrary in cases where the seismometer undergoes a pure translational displacement. The displacement of each location on the seismometer's frame then is the same as well as the ground's displacement (Fig. 1). In the presence of rotations this is no longer the case. Each point on the extended body undergoes a different displacement while the rotation is the same for all (Fig. 2). Then a reference location must be defined to which the output signal is attributed. This reference location needs not be the hinge of the pendulum nor needs it be the location of the center of mass, necessarily.

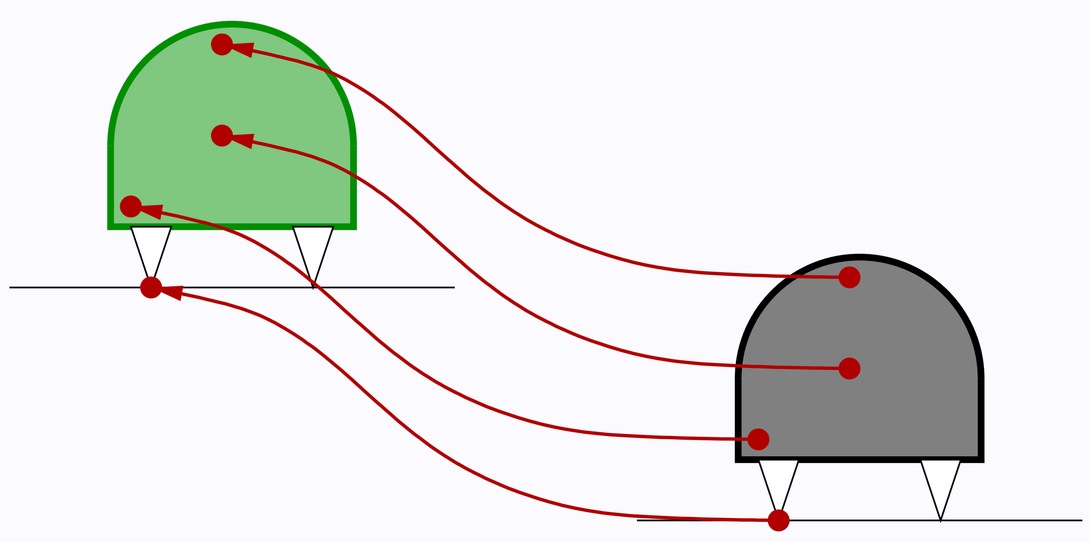

Figure 1: In the absence of rotations all locations in the seismometer's reference frame undergo the same displacement. It is not necessary to define a reference location to which the motion is attributed.

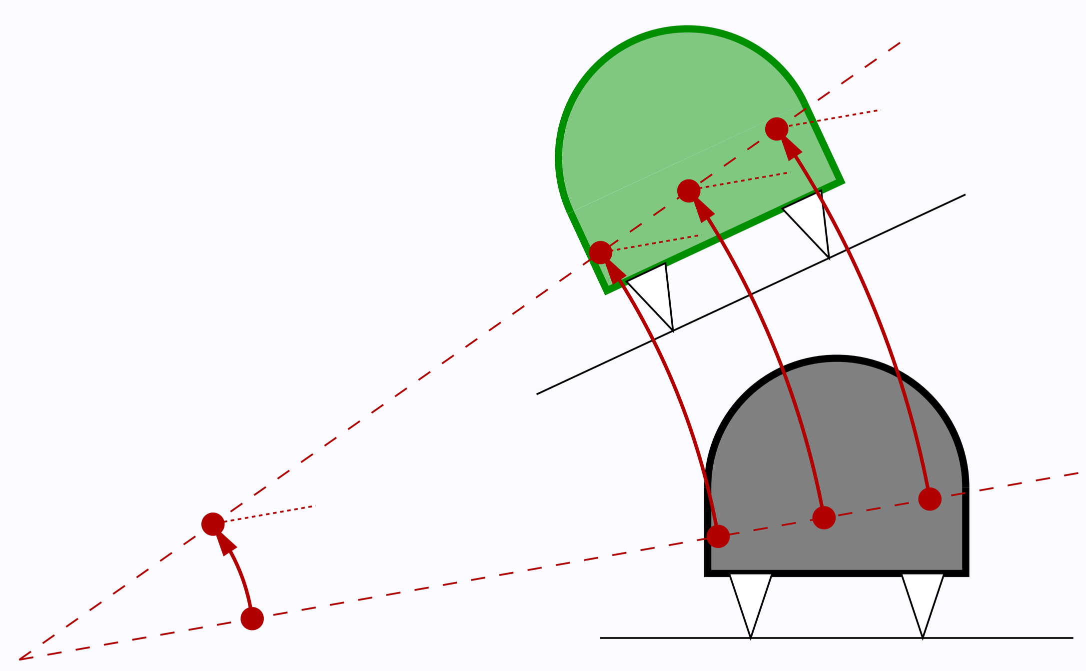

Figure 2: In the presence of rotations all locations in the seismometer's reference frame undergo a different displacement. To quantify displacement, a reference location must be defined. The angle of rotation about this reference location is independent of the choice made. Translational displacement depends on the reference location and vanishes if the center of rotation is selected as reference.

Reference location

In this study I discuss the output signal of a pendulum seismometer in response to the motion of its frame. I use a simple model of the seismometer consisting of a casing (green rectangle in Fig. 3), a pendulum (blue body) with its center of mass at S and being attached to the casing by a hinge at H. I study the response of the seismometer to a motion of the reference location R on the seismometer's frame.

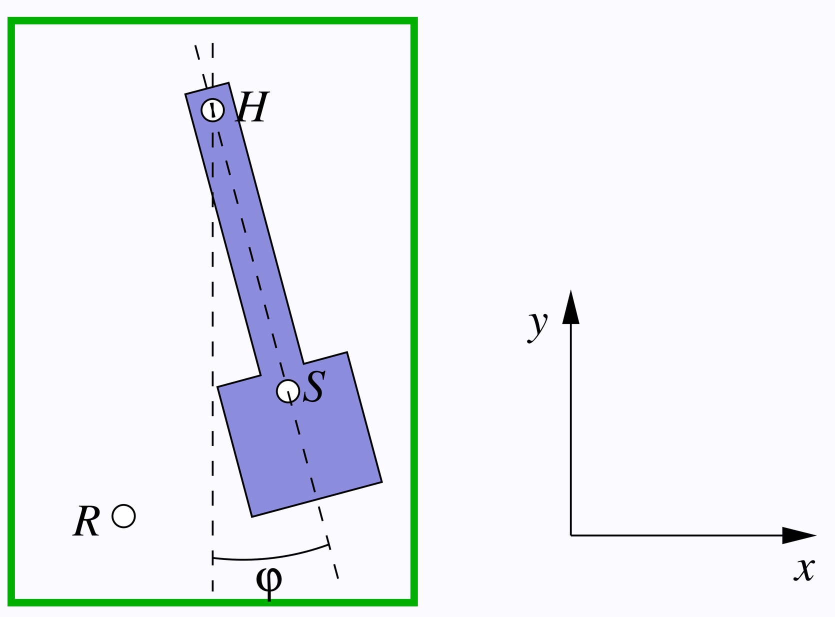

Figure 3: A simple model for a single seismic sensor. The green rectangle represents the frame of the seismometer that moves rigidly with the ground. The blue body represents the seismometer's pendulum which constitutes the seismic mass with finite moment of inertia. The center of mass is located at S. The pendulum is attached to the frame by a hinge at H which constrains the motion of the pendulum to a single degree of freedom, i.e. a rotation centered on H. R defines the reference location to which motions of the seismometer's frame are referred to. Without loss of generality all motions are restricted to the (x,y)-plane.

Referring to the point mass of the equivalent simple pendulum

Fig. 4a is a sketch of the pendulum seismometer with the reference location R placed at the location of the point mass of the equivalent simple pendulum. The equivalent simple pendulum has the whole mass of the suspended body concentrated in a point mass and has the same free period in a gravity field. This concept is known as 'the reduced pendulum' in the theory of the reversible pendulum (Rodgers 1969; Leybold 2007). The location of the point mass of the equivalent simple pendulum is sometimes referred to as 'center of oscillation' (Byerly 1952). The point mass of the equivalent simple pendulum is displaced from the hinge at H by

$$l_{\mathrm{esp}} = l + \frac{J_S}{ml} \qquad (1)$$

along the line connecting H and the center of mass at S. Here l is the distance between H and S, m is the mass of the pendulum and J_S is its angular momentum for rotations about S.

When turning the seismometer about this location, there will be no deflection of the pendulum with respect to the frame. The seismometer consequently is insensitive for rotations about this location and will only sense translational displacement of the location of the point mass of its equivalent simple pendulum. This can be understood by simple physical considerations when balancing the desire to preserve angular and linear momentum at the same time for rotations about locations on the line connecting H and S. This result will also be rigorously derived below from the pendulum's equation of motion.

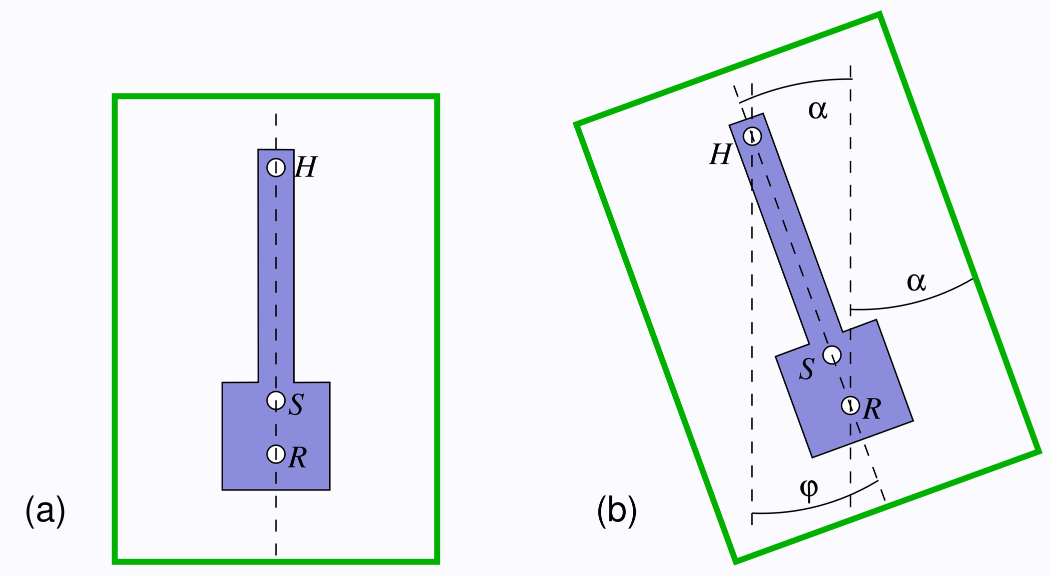

Figure 4: Rotary motion of the seismometer. a) The seismometer in its reference position with the pendulum in its rest position. b) Motion of the seismometer and pendulum in the special case of a rotation of the frame by an angle α centered on the location of the point mass of the equivalent simple pendulum. It turns out that $\ddot{\varphi} = \ddot{\alpha}$ if the distance $\overline{HR}$ equals the length of the equivalent simple pendulum.

Equation of motion

The geometrical quantities used for the mathematical treatment are defined in Fig. 5. The Lagrangian of the seismometer's pendulum is

$$L = \frac{1}{2} m \left(\dot{s}_x^2 + \dot{s}_y^2\right) + \frac{1}{2} J_S \dot{\varphi}^2, \qquad (2)$$

where s_x and s_y are components of the location vector

$$\mathbf{s}(t) = \mathbf{r}(t) - a\hat{\mathbf{f}}_1(\alpha(t)) + l\hat{\mathbf{p}}_{\parallel}(\varphi(t)) \qquad (3)$$

to the center of mass at S and $\mathbf{r}$ is the location vector to the reference location R to which the output signal is attributed. Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \qquad (4)$$

results in the equation of motion

$$ml\left(\ddot{\mathbf{r}}\,\hat{\mathbf{p}}_{\perp} - a\ddot{\alpha}\hat{\mathbf{f}}_2\hat{\mathbf{p}}_{\perp} + a\dot{\alpha}^2\hat{\mathbf{f}}_1\hat{\mathbf{p}}_{\perp}\right) + \left(ml^2 + J_S\right)\ddot{\varphi} = 0 \qquad (5)$$

after a few pages of calculus. With

$$\beta = \varphi - \alpha, \qquad \hat{\mathbf{f}}_2\hat{\mathbf{p}}_{\perp} = \cos\beta, \qquad \text{and } \hat{\mathbf{f}}_1\hat{\mathbf{p}}_{\perp} = -\sin\beta \qquad (6)$$

I obtain

$$\underbrace{\left(ml^2 + J_S\right)}_{=J_H}\ddot{\beta} = -ml\,\ddot{\mathbf{r}}\,\hat{\mathbf{p}}_{\perp} + ml\,\ddot{\alpha}\left(\underbrace{a\cos\beta}_{=a_{\parallel}} - \underbrace{\left(l + \frac{J_S}{ml}\right)}_{=l_{\mathrm{esp}}}\right) + ml\,\dot{\alpha}^2\underbrace{a\sin\beta}_{=a_{\perp}}, \qquad (7)$$

where β is the quantity observed by the transducer.

References

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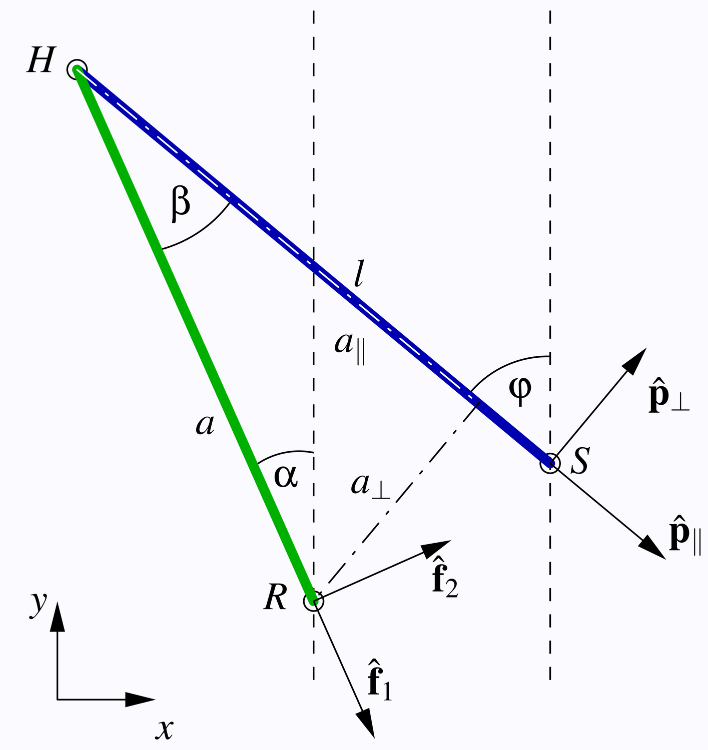

Figure 5: The geometry of the seismometer is defined by the locations of the center of mass of the pendulum's body at S, the hinge at H, and the reference point R on the frame. The response of the seismometer's pendulum to a translational displacement of R and a rotation of the frame centered on R is discussed. With respect to inertial space, φ defines the orientation of the pendulum and α defines that of the seismometer's frame. $\beta = \varphi - \alpha$ is the angle that is observed by the transducer in the seismometer. Distances $l = \overline{HS}$ and $a = \overline{HR}$ are constant and are displayed by thick lines. The unit vector in direction of the line connecting H and R is $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$ is the unit vector perpendicular to it. They are base vectors of a coordinate system that moves and turns with the frame. Similarly $\hat{\mathbf{p}}_{\parallel}$ and $\hat{\mathbf{p}}_{\perp}$ are unit vectors parallel and perpendicular to the line connecting H and S. They are base vectors of a coordinate system that moves and turns with the pendulum. Components $a_{\parallel}$ and $a_{\perp}$ of the vector from H to R in the pendulum's coordinate system are displayed by dash-dotted lines and are parallel and perpendicular to $\hat{\mathbf{p}}_{\parallel}$ respectively.

Sensitivity

I understand the pendulum to be held in its reference position by a feedback such that $a_{\parallel}$ and $a_{\perp}$ are constant. By solving eq. (7) for β I derive the expression

$$\ddot{\beta} = -\frac{1}{l_{\mathrm{esp}}}\left(\overset{①}{\ddot{r}_{\perp}} + \overset{②}{\ddot{\alpha}\left(l_{\mathrm{esp}} - a_{\parallel}\right)} - \overset{③}{\dot{\alpha}^2 a_{\perp}}\right) \qquad (8)$$

for the sensitivity of the pendulum. The three terms on the right-hand-side of eq. (8) are the contributions for three types of motion.

① translational acceleration

$\ddot{r}_{\perp}$ is the amount of translational acceleration of R in the sensitive direction $\hat{\mathbf{p}}_{\perp}$ of the pendulum. In the absence of rotary motion ($\ddot{\alpha} = 0$ and $\dot{\alpha} = 0$) it may be replaced by the linear acceleration of any other location on the seismometer's frame, e.g. $\ddot{h}_{\perp}$ for the hinge. This is the way seismometers usually are understood (by ignoring rotations). If $\mathbf{g}$ is the vector of gravity, then the effect of ground tilt can seamlessly be introduced into eq. (8) when replacing $\ddot{r}_{\perp}$ by $\ddot{r}_{\perp} - \mathbf{g}\hat{\mathbf{p}}_{\perp}$.

② angular acceleration

$\ddot{\alpha}$ is the angular acceleration of the seismometer's frame. It is scaled by $(l_{\mathrm{esp}} - a_{\parallel})$. Hence its contribution vanishes for $a_{\parallel} = l_{\mathrm{esp}}$.

③ centripetal acceleration

$-\dot{\alpha}^2 a_{\perp}$ is the centripetal acceleration acting due to a rotation about R. This contribution obviously vanishes for $a_{\perp} = 0$.

Conclusions

If the output signal is referred to the location of the point mass of the equivalent simple pendulum, i.e. R is chosen such that $a_{\parallel} = l_{\mathrm{esp}}$ and $a_{\perp} = 0$, then

$$\ddot{\beta} = -\frac{\ddot{r}_{\perp}}{l_{\mathrm{esp}}} \qquad (9)$$

for any motion. Hence the seismometer is only sensitive to the translational acceleration of the location of the point mass of the equivalent simple pendulum. When ignoring the rotary component of motion and using eq. (9) to understand the output signal of the conventional seismometer we ignore that its sensitive axis $\hat{\mathbf{p}}_{\perp}$ points to a different direction when being rotated. The relative error in the interpretation of the output signal is

$$\frac{\Delta\ddot{r}_{\perp}}{\ddot{r}_{\perp}} = \cos\Delta\alpha - 1 - \frac{\ddot{r}_{\parallel}}{\ddot{r}_{\perp}}\sin\Delta\alpha \qquad (10)$$

and can safely be ignored, since in most cases $\Delta\alpha \ll 1$. A sensor for rotary motion must be constructed such that H and S coincide. Then always

$$\ddot{\beta} = -\ddot{\alpha}. \qquad (11)$$

Acknowledgements

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